We're following Godement Chapter 3, Sections 6 & 7.

$G = GL_2$ / number field k

Suppose we have

- π_v irreducible admissible rep of local Hecke algebra $\mathcal{H}_v$, for all places v, such that π_v contains the identity rep of K_v for almost all v
- χ character of $\mathbb{A}_k^\times / k^\times$

  $\chi = \prod_v \chi_v$, χ_v character of $k_v^\times$, unramified for almost all v

Define

$$L_\pi(\chi, s) = \prod_v L_{\pi_v}(\chi_v, s)$$

formally! (there's no π yet)

The local factors L_{π_v} are described in the previous chapters of Godement's book:

$s' = 2s - \frac{1}{2}$

$$\begin{cases} 1 & \text{if } v \text{ is non-arch } \& \ \pi_v \text{ is supercuspidal} \\ L(\mu\cdot\chi_v, s') & \text{if } v \text{ is non-arch } \& \ \pi_v = \pi_{\mu,\nu}, \ \mu(x) = |x| \\ L(\nu\cdot\chi_v, s') & \text{if } v \text{ is non-arch } \& \ \pi_v = \pi_{\mu,\nu}, \ \mu(x) = |x|^{-1} \\ \boxed{L(\mu\cdot\chi_v, s')\, L(\nu\cdot\chi_v, s') \quad \text{if } v \text{ is non-arch } \& \ \pi_v = \pi_{\mu,\nu} \text{ principal series}} & \\ L(\mu\cdot\chi_v, s')\, L(\nu\cdot\chi_v, s') & \text{if } v \text{ arch } \& \ \pi_v = \pi_{\mu,\nu} \text{ principal series} \\ (2\pi)^{r-s'-t}\, \Gamma(s'+t-r) & \text{if } v \text{ arch } \& \ \pi_v = \sigma_{\mu,\nu} \text{ discrete series}, \ \mu(x) = |x|^t, \ \chi(x) = |x|^r \end{cases}$$

Let S = finite set of places $\supset$ {archimedean places}
and such that for all $v \notin S$:

- π_v contains the identity representation of K_v
- $\mathrm{Ker}(\tau_v) = \mathcal{O}_{k_v}$
- χ_v is unramified.

Write $(\varpi) = \mathfrak{m}_{k_v}$
$q = \#(\mathcal{O}_{k_v}/\mathfrak{m}_{k_v})$

Then $\pi_v = \pi_{\mu,\nu}$ is principal series and

$$L_{\pi_v}(\chi_v, s) = L(\mu - \chi_v, s')L(\nu - \chi_v, s')$$

$$= \frac{1}{1 - \frac{\mu}{\chi_v}(\varpi)q^{-s'}} \cdot \frac{1}{1 - \frac{\nu}{\chi_v}(\varpi)q^{-s'}}$$

If we assume that π_v is a pre unitary representation of $\mathcal{H}_v$,
then $|\mu(\varpi)| = q^{-\sigma_v/2}$ & $|\nu(\varpi)| = q^{\sigma_v/2}$, $0 \le \sigma_v \le 1$

so

$$L_\pi(\chi, s) = \underbrace{\prod_{v \in S}(\ \ldots\ )}_{\text{finite product}} \cdot \underbrace{\prod_{v \notin S}\left(1 - \chi_v(\varpi)q^{-s' - \frac{\sigma_v}{2}}\right)^{-1}\left(1 - \chi_v(\varpi)q^{-s' + \frac{\sigma_v}{2}}\right)^{-1}}_{\text{converges for } \mathrm{Re}(s) \gg 0}$$

Note also that for $v \notin S$ we have

$$\varepsilon_{\pi_v}(\chi_v, s) = 1.$$

This means that we get a finite product

$$\varepsilon_\pi(\chi, s) = \prod_v \varepsilon_{\pi_v}(\chi_v, s) = \prod_{v \in S} \varepsilon_{\pi_v}(\chi_v, s)$$

Recall: unitary representation of $G(\mathbb{A})$ on $L^2_0(G(k)\backslash G(\mathbb{A}), \omega)$.
If π is an irreducible component, for each place v we denote by π_v the corresponding irreducible admissible rep of $\mathcal{H}_v$.
This matches the previous setup, so for any char χ of $\mathbb{A}^\times/k^\times$ we have $L_\pi(\chi, s)$ defined for $\mathrm{Re}(s) \gg 0$.

Theorem 4: $L_\pi(\chi, s)$ is entire, bounded in every vertical strip, and satisfies the functional equation
$$L_\pi(\chi, s) = \varepsilon_\pi(\chi, s) L_\pi(\omega - \chi, 1-s).$$

Compare: for a Hecke L-function we have
$$L(\chi, s) = \prod_v L(\chi_v, s)$$
functional equation:
$$L(\chi, s) = \varepsilon(\chi, s) L(\chi^{-1}, 1-s).$$

Thursday 13 May: Following closely Section 7, Chap 3 in Godement.

$\left(W_v^0 \Big|_{K_v} = 1. \right.$ max compact.$\Big)$

$W_v \in \mathcal{W}(\pi_v)$ with $W_v = W_v^0$ for almost all v.

$W(g) = \prod_v W_v(g_v)$ $\in$ Whittaker space of the rep $\bigotimes_v \pi_v$ of global Hecke algebra $\mathcal{H}_{\mathbb{A}}$.

$\Rightarrow \exists$ K-finite function φ s.t.

$$\varphi(g) = \sum_{\xi \in k^\times} W\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

$$L_\varphi(g;\chi,s) = \int_{\mathbb{A}^\times/k^\times} \varphi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(x)^{-1} |x|^{2s-1} d^\times x$$

Note that:

- $\varphi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right)$ is rapidly decreasing as $|x| \to \infty$
- $\varphi\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} g\right) = \varphi\left(\underbrace{\begin{pmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}}_{\in Z(\mathbb{A})} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} g\right) = \omega(x^{-1}) \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} g\right)$

$$= \omega(x^{-1}) \varphi\left(\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\in G(k)} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g\right) = \underbrace{\omega(x^{-1})}_{\in S^1} \varphi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g}_{g'}\right)$$

as ω is unitary

so $\varphi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right)$ is rapidly decreasing as $|x| \to 0$.

$\Rightarrow L_\varphi(g;\chi,s)$ is entire and bounded on vertical strips.

$$L_\varphi(g;\chi,s) = \int_{\mathbb{A}^\times/k^\times} \left(\sum_{\xi\in k^\times} W\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} g\right)\right) \chi(x)^{-1} |x|^{2s-1} d^\times x$$

$$= \int_{\mathbb{A}^\times} W\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} g\right) \chi(y)^{-1} |y|^{2s-1} d^\times y$$

(where this integral converges)

$W\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} g\right)$ is rapidly decreasing as $|x| \to \infty$ and bounded (b/c it's a Fourier coeff of a cusp form)

So the last integral converges for $\mathrm{Re}(s) >> 0$.

$$\mathbb{A}^\times = \bigcup_{T \text{ finite}} \mathbb{A}^\times_T = \bigcup_{T \text{ finite}} \prod_{v\in T} k_v^\times \times \prod_{v\notin T} \mathcal{O}_{k_v}^\times$$

$$L_\varphi(g;\chi,s) = \varinjlim_T \int_{\mathbb{A}^\times_T} W\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} g\right) \chi(y)^{-1} |y|^{2s-1} d^\times y$$

$$= \varinjlim_T \prod_{v\in T} \int_{k_v^\times} W_v\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} g_v\right) \chi_v(y)^{-1} |y|_v^{2s-1} d^\times y$$

$$\cdot \prod_{v\notin T} \int_{\mathcal{O}_{k_v}^\times} W_v\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} g_v\right) \chi_v(y)^{-1} |y|_v^{2s-1} d^\times y$$

For a large enough set of places T we have

$$W_v\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} g_v\right) = 1 \qquad \text{for all } v \notin T$$

$$\chi_v(y) = 1 \qquad \text{and all } y \in \mathcal{O}_{k_v}^\times .$$

$$|y|_v = 1$$

So
$$L_\varphi(g; \chi, s) = \lim_T \prod_{v \in T} L_{W_v}(g_v; \chi_v, s)$$

But

$$L_{W_v}(g_v; \chi_v, s) = L_{\pi_v}(\chi_v, s) \quad \text{for all but finitely many } v$$

and

$$\prod_v L_{\pi_v}(\chi_v, s) \quad \text{converges for } Re(s) \gg 0.$$

$$\Rightarrow \; L_\varphi(g; \chi, s) = \prod_v L_{W_v}(g_v; \chi_v, s) \quad \text{for } Re(s) \gg 0$$

Can choose W_v for all v so that

$$L_{W_v}(e; \chi_v, s) = L_{\pi_v}(\chi_v, s)$$

e: identity element of the group

This follows from the classification of local L-factors in non-arch. and arch. cases, together with the description of the unramified case.

So for $\mathrm{Re}(s) >> 0$ we have

$$L_\pi(\chi,s) = \prod_v L_{\pi_v}(\chi_v,s) = \prod_v L_{W_v}(e;\chi_v,s) = \underbrace{L_\varphi(e;\chi,s)}_{\text{entire and bounded on vertical strips}}$$

This gives the desired analytic continuation of $L_\pi(\chi,s)$.

For the functional equation, we have

$$\frac{L_\varphi(g;\chi,s)}{L_\pi(\chi,s)} = \underbrace{\prod_v \frac{L_{W_v}(g_v;\chi_v,s)}{L_{\pi_v}(\chi_v,s)}}_{\text{all but finitely many factors are 1}}$$

and

$$\frac{L_\varphi(wg;\omega-\chi,1-s)}{L_\pi(\omega-\chi,1-s)} = \prod_v \frac{L_{W_v}(wg_v;\omega-\chi_v,1-s)}{L_{\pi_v}(\omega-\chi_v,1-s)}$$

$$\overset{\text{local functional eq.}}{=} \prod_v \varepsilon_{\pi_v}(\chi_v,s)\frac{L_{W_v}(g_v;\chi_v,s)}{L_{\pi_v}(\chi_v,s)}$$

$$= \varepsilon_\pi(\chi,s)\frac{L_\varphi(g;\chi,s)}{L_\pi(\chi,s)}$$

Finally:

$$L_\varphi(wg;\omega-\chi,1-s) = \int_{\mathbb{A}^\times/k^\times} \varphi\left(\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} wg\right)\omega^{-1}\chi(x)\,|x|^{1-2s}\,d^\times x$$

$$\overset{\varphi(\omega^{-1}h)=\varphi(h)}{=} \int_{\mathbb{A}^\times/k^\times} \varphi\left(\underbrace{\omega^{-1}\begin{pmatrix} x^{-1} & 0 \\ 0 & x^{-1}\end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\omega}_{\begin{pmatrix} x^{-1} & 0 \\ 0 & 1\end{pmatrix}} g\right) \chi(x)|x|^{1-2s} d^\times x$$

$$= \int_{\mathbb{A}^\times/k^\times} \varphi\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & 1\end{pmatrix} g\right) \chi(x)|x|^{1-2s} d^\times x$$

$$\overset{y=x^{-1}}{=} \int_{\mathbb{A}^\times/k^\times} \varphi\left(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} g\right) \chi(y)^{-1}|y|^{2s-1} d^\times y$$

$$= L_\varphi(g; \chi, s)$$

We conclude that

$$L_\pi(\chi, s) = \varepsilon_\pi(\chi, s) L_\pi(\omega - \chi, 1-s).$$